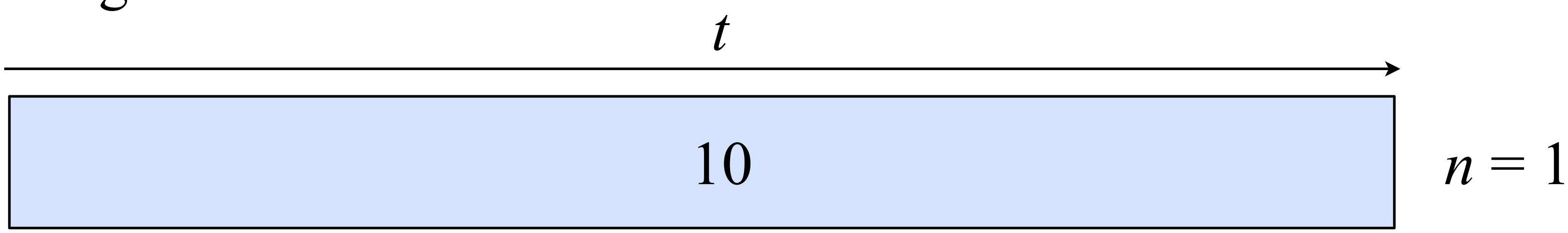
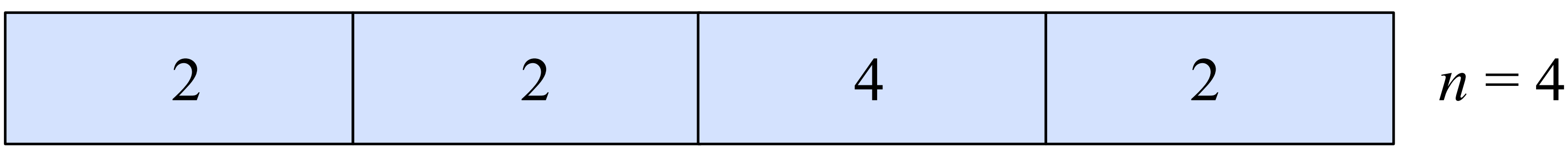
Poisson probabilities

Over a branch that has been in existence for some time period *t*, imagine that there have been 10 events...



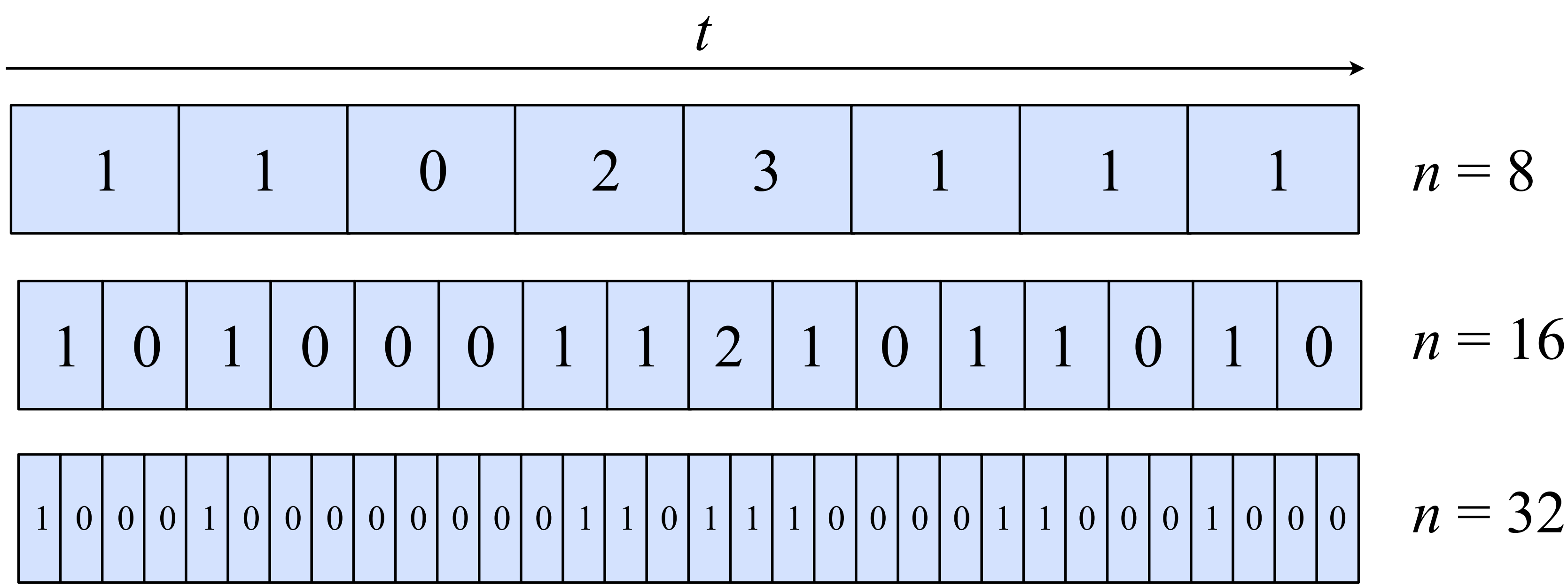
Now divide the time interval by 2...

Then divide it by two again...



Poisson probabilities

Keep on dividing it until no interval has more than 1 event...



At this point there are only two categories of intervals:

$y = 10$ intervals contain 1 event

$n - y = 22$ intervals contain 0 events

Poisson probabilities

1	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	0	0	0	0	1	1	0	0	0	1	0	0	0

$n = 32$

We can treat this as a series of n = 32 independent trials, each of which resulted in either a success (1) or a failure (0).

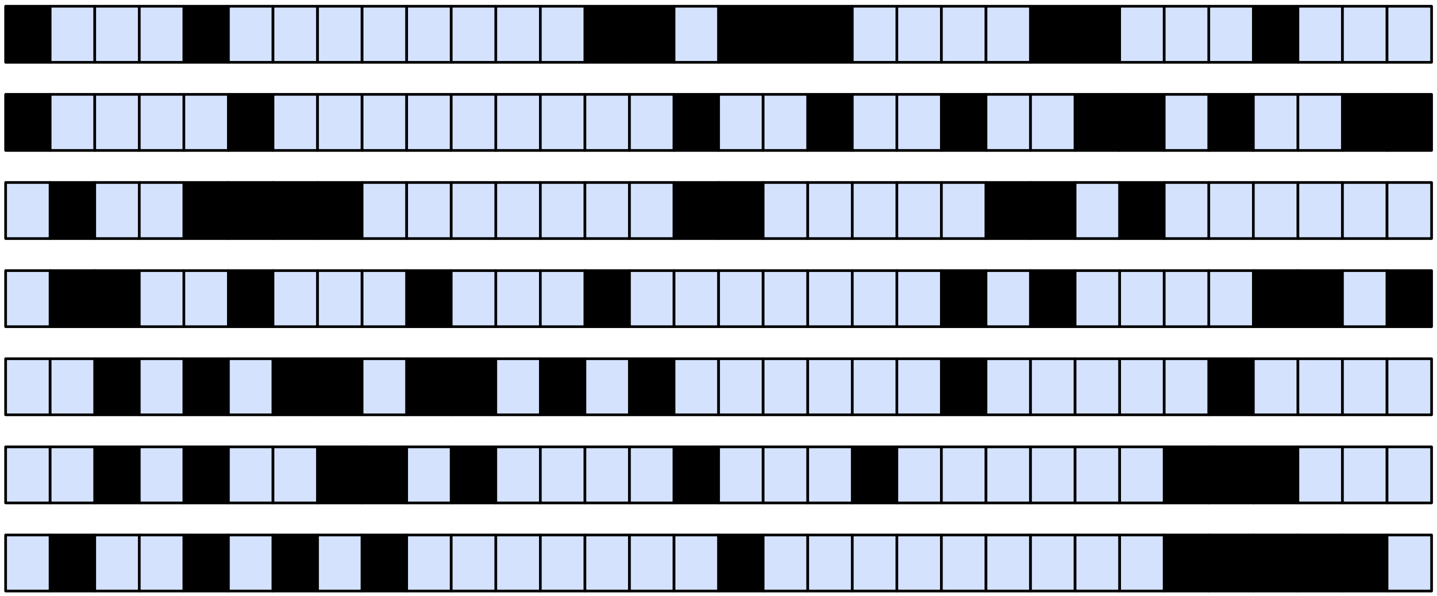
If we let the probability of success on each trial be *p*, the probability of failure is 1 - *p,* and the probability of 10 events is given by the binomial probability formula:

$$\Pr(y|p) = \binom{n}{y} p^y (1-p)^{n-y}$$

where the first term is the number of ways of rearranging the *y* 1s amongst the *n* bins (see next slide).

Poisson probabilities

Here are a few examples of configurations each involving y=10 successes in n=32 trials, but there are actually many, many more (see below).



The formula below provides the number of different ways that y=10 univents could be distributed amongst n=32 bins.

$$\binom{n}{y} = \frac{n!}{y!\,(n-y)!} = \frac{32!}{10!\;22!} = 64512240$$

See http://en.wikipedia.org/wiki/Factorial if you are unfamiliar with factorials.

Poisson probabilities

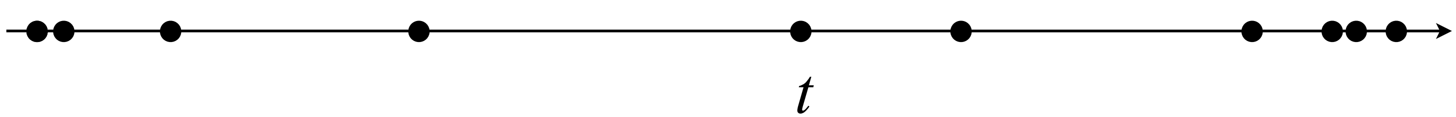
Now imagine continuing to **double** the number of bins while **halving** *p* so that the expected total number of events (λ) stays constant. This amounts to keeping the product of *n* and *p* constant:

$$\lambda = np = (2n)\left(\frac{p}{2}\right)$$

Replacing *p* with the equivalent value λ/*n* and finding the limit as *n* is increased yields

$$\Pr(y|\lambda) = \lim_{n\to\infty}\left\{\left(\frac{n!}{y!\,(n-y)!}\right)\left(\frac{\lambda}{n}\right)^{y}\left(1-\frac{\lambda}{n}\right)^{n-y}\right\}$$

We have now divided our time *t* into infinitely many bins, and the intervals in which a univent has occurred appear as points along a continuous time line:



Poisson probabilities

$$\Pr(y|\lambda) = \lim_{n\to\infty} \left(\frac{n!}{y!\,(n-y)!}\right)\left(\frac{\lambda}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^{n-y}$$

$$= \frac{\lambda^y}{y!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-y}\left(\frac{n!}{n^y\,(n-y)!}\right)$$

Both of these go to 1 as n becomes very large

This term goes to $e^{-\lambda}$ as n becomes very large

Poisson probability formula:

$$\Pr(y|\lambda) = \frac{\lambda^y}{y!}e^{-\lambda}$$

Important special cases:

$$\Pr(y=0|\lambda) = e^{-\lambda}$$
$$\Pr(y>0|\lambda) = 1-e^{-\lambda}$$

The quantity λ is the expected number of events across the interval of length t. We can thus replace λ by the event rate μ times the time t: $\lambda = \mu\, t$

So how did *e* sneak in there?

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Think of the symbol *e* as just a shorthand way of writing the limit on the right side of the equation. The above equation is in fact the *definition* of *e*. (The limit on the previous page involves powers of *e*, but note that setting the power to 1 returns you to the equation above.)

You can easily verify the truth of the equation by plugging in larger and larger values of *n*:

$$\begin{aligned} n = 10 \quad & (1 + 1/10)^{10} = 2.594 \\ n = 100 \quad & (1 + 1/100)^{100} = 2.705 \\ n = 1000 \quad & (1 + 1/1000)^{1000} = 2.717 \\ n = 10000 \quad & (1 + 1/10000)^{10000} = 2.718 \end{aligned}$$

For comparison, the constant *e* = 2.718... (the dots indicate that the digits go on forever; i.e. *e* is an irrational number)