Scientific units as roadmaps and spies: a guide to units manipulation for undergraduates.

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Chapter 1: INTRODUCTION

1.1 What ARE units, anyway?

So what are units, really? We think of inches, of centimeters, or of teaspoons or miles per hour, but where do these measures come from and what good are they? First, realize that ***we*** have invented units for our own convenience and communication. Many had their humble beginnings as really quite arbitrary measures, equaling the length of one king's thumb, or the amount of light from one candle. Since most people's thumbs are somewhat the same size, saying a brick is three thumb-lengths long communicates some information, but it certainly isn't exact. Nowadays, units such as lengths, weights, amounts of time, and so on are defined according to internationally agreed-upon standards, and many of these standards are based on things that can be observed almost anywhere (such as the distance light can go in a certain amount of time).

Not all units, though, have arbitrary or imprecise origins. Some units are more "natural"--for example one day or one year--and some are constant in our universe...for example, the amount of charge carried by an electron, designated "e." The Fahrenheit and Celsius temperatures at which pure water freezes or becomes steam were arbitrarily defined as 32°F and 212°F, and 0°C and 100°C, but absolute zero, 0° Kelvin, is truly an absolute physical limit for temperature. There is no negative Kelvin temperature in the universe as we know it.

Who defined the units we so often use today, and why? For some units, we can answer that question (see http://www.mel.nist.gov/div821/museum/timeline.htm for a brief timeline of the history of the meter). For others, we can't. However, units are standardized today, and we can bring out a ruler or thermometer or scale and using units we can know exactly how much of something we have in our shopping bag, exactly how thick the insulation is in our house, or exactly how worn the treads in our tires are.

1.2 Some everyday examples of units:

Have you ever stood in the produce department of your local grocery store and wondered whether the loose peaches, labeled $1.59/lb, are more or less expensive than the prewrapped nectarines ($2.00 for a 20 oz tray) that you really really want? Or, have you planned a trip to Canada, checked the web, seen that it's 25 °C in Quebec, and wondered whether you should pack your parka or your bathing suit? Or what about baking cookies in your new apartment, when the only measuring spoon you can find is 1 teaspoon, and you need a tablespoon of butter? All these are common examples of units we often use, and units conversions we need to make in order to function in daily life. Where did these units come from? Again, they may have arbitrary beginnings hidden in the depths of history, but we agree upon them now, and they make it possible to quantify, to communicate, and to repeat procedures.

1.3 Making the leap--units in the classroom:

We're more or less comfortable with particular units and conversions depending on our backgrounds; if you are from the USA, you're likely to recognize miles per hour, inches, and temperature Fahrenheit much more readily than kilometers per hour, centimeters, and temperature in Celsius. Since all modern units are referenced to standards, there is no difference in length between something 1 inch long and something 2.5400 cm long, but for most of us, one description contains much more intuitive information than the other. This same bias follows us to the classroom; some units we learn in chemistry or physics or biology or psychology are intuitive and comfortable. Some, though, are completely new, and they may be combined in ways that simply don't make sense to us initially. Sometimes units seem to appear from nowhere, or in order to solve some problem, they have to be turned upside down and divided or multiplied in ways that seem artificial or magical. This book is meant to help you to see that instead of being confusing and frustrating, units can actually be extremely useful. We'll illustrate how units can:

- help you convert from nonintuitive to intuitive measures
- guide you in manipulating data to arrive at some defined goal
- assist you in figuring out what key information you don't have for the calculation you need to do
- help you to assess whether you took an appropriate approach in solving a problem, and whether the calculations you did make sense.

In short, units are your friends!

End of chapter problems:

1. Name as many units as you can. Think through your daily life....driving your car, buying lunch, sitting in classes, reading the paper, listening to the news.

2. Think through your activities today, and write down what units conversions you used or wondered about.

CHAPTER 2--Units gymnastics, and units as roadmaps for conversions

You are finally driving in Canada for your vacation, and it is 20 kilometers to the next town. You're very hungry, it's almost lunchtime...your cousin wonders out loud how many miles are in 20 kilometers. Answering this kind of question can be tricky, but interestingly enough you can solve it by multiplying by "1"! Granted, you will use "1" in a special form, but it will be "1" none-the-less.

2.1 A key to using units...multiplying by "1":

In the next sets of examples, you are going to see the many useful ways you can multiply by "1" in an equation in order to convert from one set of units to another. This is a key step toward understanding how units are manipulated during all sorts of calculations.

Let's think first about distances. A kilometer is a human-defined unit of a certain length, and a mile is human-defined unit of a different length. But the actual distance between two points is the same, whether expressed in kilometers or miles, French, English, or Russian. Because this is true you can represent a statement such as "there are 0.62 miles in a kilometer" mathematically as:

$$0.62\,miles = 1kilometer$$

So, you can divide both sides of the equation above by 1 kilometer to get:

$$\frac{0.62miles}{1kilometer} = \frac{1kilometer}{1kilometer} = 1$$

When you need to convert from miles to kilometers, you can use the small equation

$\frac{0.62miles}{1kilometer} = 1$ to do it! How? Let's look at an example........

Example 1: You are driving in Canada, and it is 20 kilometers to the next town. How far is this in miles?

First, you need to find information telling you how many miles there are in a kilometer. You might look in the back of this book, and see that there are about 0.62 miles in a kilometer. So, how do you set up this problem? First, translate the words you are using in your head into a mathematical expression. To express the information that you just gathered, that "there are 0.621 miles in a kilometer," you could write:
1 kilometer = 0.62 miles

You have the information that it is 20 km to the next town. To express the question you need to answer, you might write:

20 kilometer = ? miles

Intuitively, you probably know you need to multiply or divide 20 by 0.62 in order to get your answer. But which is it? Units will help you know this! Rewrite the problem you have in the following way, putting the units of the answer you want on the right side of the equation, and expressing your starting information on the left:

$$(20\,kilometer) * \left(\frac{0.62\,miles}{1\,kilometer}\right) = ?\,miles$$

Notice that you can cancel out the unit kilometer, just the way you could cancel out an "A" or a "B" in Box 2. After canceling on the left hand side of the equation, you end up with "miles" as the unit, matching perfectly with the units you ultimately want (on the right hand side of the equation)!

$$(20) * \left(\frac{0.62\,miles}{1}\right) = (20 * 0.62)\,miles = 12.4\,miles$$

Now "wait just one minute" you say....here's a perfect example of units appearing out of nowhere, apparently just because we want to be able to cancel out kilometers. WHY IS THIS FAIR??? It's fair because, as we hinted before, all we're doing on the left hand side of this equation

$$(20\,kilometers) = ?\,miles$$

is multiplying by "1", in this way:

$$(20\,kilometer) * 1 = (20\,kilometer) * \left(\frac{0.62\,miles}{1\,kilometer}\right) = ?\,miles$$

True, we've written "1" in a special form , but it's "1" nonetheless. You've simply inserted a new representation of "1" that takes into account differences between our arbitrarily defined units miles and kilometers, and this allows you to convert from a known distance expressed in kilometers to the same known distance expressed in miles.

This last example might seem to make a very simple conversion extremely difficult! Have no fear! Soon you will be multiplying by "1" in equations without even worrying about it, and you'll have an intuitive feel for what representation of "1" you might want to use. For example, what if you knew the distance from your car to the town in miles, and you wanted to know the distance in kilometers in order to tell your French cousin in units she is used to? How would you set up the problem?

Example 2: Your French cousin is driving in Canada, and it is 12.4 miles to the next town. How far is this in kilometers?

You already know there are about 0.62 miles in a kilometer. So, how do you set up this problem? As before, first, write down the information you know:

1 kilometer = 0.62 miles

You have the information that it is 12.4 miles to the next town. Express the problem you need to solve mathematically:

12.4 miles = ? kilometers

Now rewrite the problem you have, putting the number you want on the right side of the equation, and expressing all your starting information on the left:

$$(12.4\,mile) * 1 = ?\,kilometer$$

or

$$(12.4\,mile) * \left(\frac{1\,kilometer}{0.62\,mile}\right) = ?\,kilometer$$

Notice that you can cancel out the unit "mile", just the way you could cancel kilometers in example 1. On the left hand side of the equation, you end up with "kilometer" as the unit, matching perfectly with what you want (on the right hand side of the equation)!

$$(12.4\,mile) * \left(\frac{1\,kilometer}{0.62\,mile}\right) = 20\,kilometer$$

This time, you have multiplied the left-hand side of the equation by "1" in the form of

$\left(\frac{1\,kilometer}{0.62\,miles}\right)$, so that you can get kilometers on the right-hand side of the equation.

Take one more look at examples 1 and 2. Notice that we are using the fact that

$\frac{1\,kilometer}{0.62\,miles} = 1 = \frac{0.62\,miles}{1\,kilometer}$. This makes sense! If you are explaining to your little brother that

there are three teaspoons in a tablespoon, you might say "there are three teaspoons in a tablespoon," or you might say "one tablespoon has three teaspoons in it." Both are of course totally correct, and you could represent them mathematically as:

$$\frac{3\,teaspoons}{1\,tablespoon} = 1 = \frac{1\,tablespoon}{3\,teaspoons}$$

2.2 Basic rules for manipulating scientific units:

These examples have been fairly straightforward so far, but sometimes conversions of units can become more complicated. Let's establish some basic rules that you can use to manipulate units. You'll find that you already know a lot of these rules; you have seen them for years in school, since learning about fractions!

Take a look at Box 1. In it, you will find a list of basic rules for manipulating fractions and exponents. This is ALL you'll need to know in order to begin flipping, flopping, choosing, and manipulating units like a pro. Notice that we've already used the "canceling out" rules in examples 1 and 2. As we go through the next set of examples, make sure you understand the rules we are using in order to manipulate fractions.

Box 1: Fraction and exponent rules

<u>Basic fraction manipulation:</u>

$$\frac{A}{B} \quad \frac{C}{D} = \frac{A*C}{B*D}$$

$$\frac{1}{\left(\frac{A}{B}\right)} = \frac{B}{A}$$

so..... $\frac{\left(\frac{A}{B}\right)}{\left(\frac{C}{D}\right)} = \left(\frac{A}{B}\right) \quad \frac{1}{\left(\frac{C}{D}\right)} = \frac{A}{B} \quad \frac{D}{C} = \frac{A*D}{B*C}$

<u>Exponents:</u>

$A^2 = A*A$, and $A^3 = A*A*A$, and so on...

$(A*B)^2 = (A*B)*(A*B) = A*B*A*B = A*A*B*B = A^2*B^2$ and so on...

$\left(\frac{A}{B}\right)^2 = \left(\frac{A}{B}\right)*\left(\frac{A}{B}\right) = \frac{A*A}{B*B} = \frac{A^2}{B^2}$, and $\left(\frac{A}{B}\right)^3 = \left(\frac{A}{B}\right)*\left(\frac{A}{B}\right)*\left(\frac{A}{B}\right) = \frac{A*A*A}{B*B*B} = \frac{A^3}{B^3}$, and so on...

Also, $A^{-1} = \frac{1}{A}$, and $A^{-2} = \frac{1}{A^2}$, and so on....

so, for example, ... $B*\left(A^{-2}\right) = B*\frac{1}{A^2} = \frac{B}{A^2}$

Finally, $A^2*A^3 = (A*A)*(A*A*A) = A*A*A*A*A = A^5$

in other words, $A^n*A^m = A^{(n+M)}$

and

$\left(A^3\right)^2 = (A*A*A)*(A*A*A) = A^6$ in other words, $\left(A^n\right)^m = A\left(^{n*m}\right)$

Canceling out:

$$\frac{A}{B} * \frac{B}{C} = \frac{A*B}{B*C} = \frac{A*\not{B}}{C*\not{B}} = \frac{A}{C}$$

so, for example,..... $\frac{A^2}{B} * \frac{C}{A} = \frac{A*\not{A}*C}{B*\not{A}} = \frac{A*C}{B}$

2.3 Steps to take when solving a problem with units conversions:

Now let's try a slightly more difficult example, where you have several pieces of information you need to incorporate into an equation in order to get the answer you need in the units you want. And let's formalize some steps you can take when preparing to solve such problems.

Example 3. You are watching the news and the announcer marvels that at burnout of the solid rocket boosters, the space shuttle is going up into space at 4250 feet per second. You wonder how fast this is in miles per hour, a more intuitive measure of speed for you.

Step 1: **Write down in shorthand the information you know**:

- The rocket is moving at 4250 ft per second.
- There are 5280 feet in a mile.
- There are 60 seconds in a minute, and there are 60 minutes in an hour.

Step 2: **Start putting your information into an equation**, expressing the problem you need to solve mathematically.

$$\frac{4250\text{feet}}{\text{second}} * ?? * ?? = \frac{?\text{miles}}{\text{hour}}$$

Step 3: Examine the units in your equation.
On the right hand side, you have miles in the numerator, and hours in the denominator. On the left hand side of the equation, you have feet in the numerator, and seconds in the denominator. You need to convert units of distance (feet to miles), and you need to convert units of time (seconds to hours). From the back of this book (or from your head), you know there are 5280 feet in a mile. Also, you know there are 60 seconds in a minute, and there are 60 minutes in an hour. In solving this problem, we can use the following representations of "1":

$$\frac{60 \text{ seconds}}{1 \text{ minute}} = 1 = \frac{1\text{minute}}{60 \text{ seconds}}$$

and

$$\frac{60\text{minutes}}{1\text{hour}} = 1 = \frac{1\text{hour}}{60\text{minutes}}$$

and

$$\frac{1 mile}{5280 feet} = 1 = \frac{5280 feet}{1 mile}$$

So, looking at how we want our equation to turn out on the right-hand side, and how the units will have to cancel out to get to those appropriate units, we can multiply the left-hand side of our equation by 1 in several forms, and get the units we need. Let's do this step by step:

$$\frac{4250 \text{feet}}{\text{second}} * 1 * 1 * 1 = \frac{? \text{ miles}}{\text{hour}}$$

First, try to get the distance in feet converted to distance in miles. You'll want to use some representation of "1" that allows you to cancel out feet, and have miles left in the numerator. Let's use:

$$\left(\frac{4250 \text{feet}}{1 \text{second}}\right) * \left(\frac{1 \text{mile}}{5280 \text{feet}}\right) * 1 * 1 = \left(\frac{4250 * 1 \text{mile}}{5280 \text{second}}\right) * 1 * 1 = ? \frac{\text{miles}}{\text{hour}}$$

So far so good; you now have miles in the numerator on the left-hand side of the equation. But, you also still have seconds in the denominator. How are you going to change seconds to hours? Once again, multiply by "1" in some form that lets you cancel out seconds and end up with hours. In this case, since you know how many seconds are in a minute, and you know how many minutes are in an hour, you can get to the units "hours" from units "seconds" using two steps:

$$\left(\frac{4250 * 1 \text{mile}}{5280 \text{second}}\right) * \left(\frac{60 \text{second}}{1 \text{minute}}\right) * \left(\frac{60 \text{minute}}{1 \text{hour}}\right) = ? \frac{\text{miles}}{\text{hour}}$$

To solve for miles per hour, you can now cancel out the units minutes and seconds, and get the following answer:

$$\left(\frac{4250 * 1 \text{mile} * 60 * 60}{5280 * 1 \text{hour}}\right) = 2898 \frac{\text{miles}}{\text{hour}}$$

Let's use one more example, this one slightly more difficult than the last.....

Example 4. You work in a candy factory, and you observe that your chocolate enrober is able to drizzle chocolate onto 45 peanut clusters per second. An enormous peanut cluster order comes in; the customer needs 10,000 peanut clusters by tomorrow at noon. If it is 4:00 PM now, can you produce the candy your customer needs by noon the next day?

Step 1: **Write down in shorthand the information you know**:

- Your machine can produce 45 peanut clusters per second.
- You have only 20 hours left before noon tomorrow.
- There are 60 seconds in a minute, and 60 minutes in an hour.

Step 2: **Start putting your information into an equation**, expressing the problem you need to solve mathematically.

You know that you want to calculate how many peanut clusters your enrober can produce in one hour. Knowing that, you can figure out how many can be finished in 20 hours, and then you want to compare that to the 10,000 your customer needs to see if you can fill the order. Start writing down your mathematical representation of the information you have and need. You may not have all the information you need yet! But if you start trying to write the equation, it will help you figure out what information you DO need to finish the equation and solve it. First, let's figure out how many peanut clusters can be made in one hour:

$$\frac{45 \text{ peanut clusters}}{\text{second}} * ?? * ?? = \frac{? \text{ peanut clusters}}{1 \text{hour}}$$

Step 3: Examine the units in your equation.

On the right hand side, you have peanut clusters in the numerator, and hours in the denominator. On the left hand side of the equation, you have peanut clusters in the numerator, but you have seconds in the denominator. You need to change seconds on the left-hand denominator into hours.....and you know how to do this! You know there are 60 seconds in a minute, and there are 60 minutes in an hour. So, looking at how we want our equation to turn out, and how the units will have to cancel out, we can multiply the left-hand side of our equation by 1 in several forms, and get the units we need:

$$\frac{45 \text{ peanutclusters}}{1 \text{ second}} * 1 * 1 = \frac{? \text{ peanutclusters}}{1 \text{ hour}}$$

Let's use:

$$\frac{45 \text{ peanutclusters}}{1 \text{ second}} * \frac{60 \text{ seconds}}{1 \text{ minute}} * \frac{60 \text{ minutes}}{1 \text{ hour}} = \frac{? \text{peanutclusters}}{1 \text{ hour}} = \frac{162000 \, peanutclusters}{1 hour}$$

$$\frac{45 \text{ peanutclusters} * 60 * 60}{1 * 1 * 1 \text{hour}} = \frac{? \text{peanutclusters}}{1 \text{ hour}} = \frac{162000 \, peanutclusters}{1 hour}$$

Notice on the left-hand side of the equation you can cancel out units of seconds and minutes, leaving you with hours in the denominator....exactly what you need! And, wow! Your enrober can produce 162,000 peanut clusters in one hour! So, in 20 hours, it can produce:

$$\frac{162{,}000 \text{ peanut clusters}}{1 \text{ hour}} * (20 \text{ hours}) = 3{,}240{,}000 \text{ peanut clusters.}$$

Clearly, you can indeed easily fill the customer's order.

2.4 Units pitfalls, and strategies for avoiding them

Notice that throughout this text, we always include a "*" to indicate "multiplied by." For example, we wrote

$$\left(\frac{1 kilometer}{0.62 \, mile}\right) * (12.4 \, mile) = 20 \, kilometer$$

You'll find that depending on your discipline, the text you are using, your teacher's preferences, or your own shorthand, the "*" may be represented by an "x", by a "·", or by nothing at all (especially if parentheses are being used to separate portions of the equation). For example, you might see the equation above represented as:

$$\left(\frac{1\,kilometer}{0.62\,mile}\right) \times (12.4\,mile) = 20\,kilometer$$

or

$$\left(\frac{1\,kilometer}{0.62\,mile}\right) \cdot (12.4\,mile) = 20\,kilometer$$

or

$$\left(\frac{1\,kilometer}{0.62\,mile}\right)(12.4\,mile) = 20\,kilometer$$

All three notations are correct; all three are common. Also, notice that we tend to enclose individual pieces of information within parentheses, at least when first setting up our equations. You should do this too, because it will help you keep those individual pieces of information separate and straight.

Pitfall #1: canceling out incorrectly

As you already undoubtedly know, many units have abbreviations that make life much easier when you are writing out long equations. As an example, minutes is abbreviated *min*, meters is abbreviated *m*, kilometers is abbreviated *km*, and inches is abbreviated *in.* If you aren't familiar with these abbreviations, and you don't enclose information in parentheses when you are manipulating and crossing out units, you can get yourself into trouble fast! First let's cancel out units correctly in the following example, and then we'll take a look at what happens if you are unfamiliar with units abbreviations, and you start writing out a problem where you have to divide minutes by meters....

Example 5: You know that it takes you 5 minutes to run 1000 meters in gym class, and you are interested in figuring out how fast you should be able to run a 10 kilometer race if you keep you your pace.

As usual, **Step 1: Write down in shorthand the information you know**:

- You run 1000 meters in 5 minutes
- The race you want to run is 10 kilometers long
- One kilometer is 1000 meters

Step 2: Start putting your information into an equation expressing mathematically the problem you need to solve.

First, you note that you know it takes 5 minutes to run 1000 meters. You need to know how long it would take to run 1 kilometer, and once you know that, you can figure out how long it would take to run your 10 kilometer race. Start with the first part of this calculation:

$$\left(\frac{5min}{1000m}\right) * "1" = \frac{?\text{min}}{1km}$$

You need to multiply by "1" in some form to get kilometers in the denominator, instead of meters. You can do this! You know 1000m=1kilometer. So...

$$\left(\frac{5min}{1000m}\right) * \left(\frac{1000m}{1km}\right) = \frac{5\text{min}}{1km}$$

On the left side of the equation, you can cancel out meters, leaving you with units that match what you want in your answer on the right-hand side of the equation!

Now that you know how many minutes it takes to run 1 kilometer, you can figure out how many minutes it takes to run 10 kilometers. Use the units to guide you.....you want to know how many minutes you'll be running in that race. Minutes is the only unit you want on the right-hand side of the equation. This means that you need to get rid of "kilometers" in the denominator of

$$\frac{5\text{min}}{1km}$$

So, you can multiply by the length of the race (in kilometers), and your units cancel out properly:

$$\left(\frac{5min}{1km}\right) * (10km) = 50\text{min}$$

Step 3: Examine the units in your equation.
If you keep the parentheses around your information, and you keep in mind what the abbreviations mean, you are in fine shape. You can cancel out *m*, and *km*, ending up with *min* in the numerator, which is exactly what you want! It will take you approximately 50 minutes to run your 10*km* race.

BEWARE! PITFALL!!

What happens, though, if you don't keep track of units, and you start trying to cancel out letters that look as though they might represent units........what if, for example, you take a look at the first term of the equation above, and forget what those abbreviations are.......

$\frac{5min}{1000m}$ .....and you try canceling out letters, because it looks as though you can......

$\frac{5\not{m}in}{1000\not{m}} \neq 0.005in$ ...and suddenly you have inches! Clearly this isn't correct!

Believe it or not, this kind of mistake is easy to make if you aren't clear on what the abbreviations for units are, and it happens especially when you are using units that are new to you. Beware! Keep track of groups of letters that represent individual units, and don't get confused by notation. Parentheses will help you to keep units separated; use them.

Pitfall #2: missing exponents in conversions of area units to other area units, volume units to other volume units, etc.

This may be one of the most common types of units errors made early on in your analytical career. As a most simple example, sit back for a moment and think about centimeters and meters. (We'll use the abbreviation cm for centimeters, and m for meters.) You probably know there are 100 centimeters in a meter. Now, off the top of your head, how many square centimeters do you think there are in a square meter? 100? 1000? 10000? Careful! Let's see which is correct....

We know: $100cm = 1m$

If we want to know how many cm^2 there are in a m^2, then we need to square each side of the equation shown above...

giving us $(100cm)^2 = (1m)^2$

Following our exponents rules in box 1,

$$(A * B)^2 = A^2 * B^2$$

so this means $(100cm)^2 = (100)^2 (cm)^2 = 10000 cm^2$

Now you know that

$$10000cm^2 = 1m^2$$

Does this surprise you? Did you think the answer was 100? Similarly, are you surprised that there are 144 square inches in one square foot? or 1728 cubic inches in a cubic foot? If you were surprised by this answer, check it over again, and make sure you understand it. Try this example problem:

Example 6. Your French cousin owns a vineyard, and she tells you its area is 4 square kilometers. How many square miles is this?

As usual, **Step 1: Write down in shorthand the information you know**:

- Your cousin's vineyard has an area of 4 km^2.
- There are 0.62 miles in a kilometer.

Step 2: Start putting your information into an equation expressing mathematically the problem you need to solve:

$$4kilometer^2 = ?\ miles^2$$

We're going to multiply the left hand side of the equation by a slightly more sophisticated version of "1" this time....we're going to use the facts that

$0.62\,mile = 1\,kilometer$,

so $(0.62\,mile)^2 = (1\,kilometer)^2$,

so $\left(\frac{(0.62\,mile)^2}{(1\,kilometer)^2}\right) = 1 = \left(\frac{0.3844\,mile^2}{1\,kilometer^2}\right)$

0

Let's use that final representation of 1 in our equation:

$$4\,kilometer^2 * 1 = 4\,kilometer^2 * \left(\frac{0.3844\,mile^2}{1\,kilometer^2}\right) = 4 * 0.3844\,mile^2 = ?\,miles^2$$

Step 3: Examine the units in your equation.

They look good! You can cancel out $kilometer^2$, and end up with $miles^2$. The area of the $4km^2$ vineyard is $1.5376mile^2$.

BEWARE! PITFALL!

The major pitfall in this kind of problem is that you forget to square everything you need to square in order to multiply by the correct representation of "1." Notice, we said:

$$\left(\frac{(0.62\,mile)^2}{(1\,kilometer)^2}\right) = 1 = \left(\frac{(0.62)^2\,mile^2}{(1)^2\,(kilometer)^2}\right) = \left(\frac{0.3844\,mile^2}{1\,kilometer^2}\right)$$

Don't fall into the trap of only squaring the units themselves (for example miles in this case), while forgetting to square the (0.62)! There are (0.62)*(0.62)=0.3844 $mile^2$ in one $kilometer^2$.

Pitfall #3: grouping and flipping fractions incorrectly

Let's illustrate one more pitfall.....the grouping and flipping fractions pitfall, then work through an example.

Imagine you are working your cousin's garden, and you realize that all around you there are tiny ant hills in the soil. Ants are milling around, pouring out of the ant hills in search of food. You decide to figure out how many ants are appearing out of these little hills over time, and you decide you'll count the number of ants coming out in just one small corner of your garden. So, you and your cousin sit down with magnifying glasses and stopwatches and you count that 150 ants come out of the ant hills over the course of one minute. You measure the size of your corner of the garden, and see that it is about one square meter in size. To describe these ant movements, you can say that 150 ants come out of ant hills every minute per square meter of garden plot. OR, you could just as easily say that 150 ants come out of a square meter of soil over the course of one minute. Mathematically, you could describe this ant movement in several ways, including:

$$\left(\frac{\left(\frac{150ants}{1\min}\right)}{1m^2}\right) \quad \text{or} \quad \left(\frac{\left(\frac{150ants}{1m^2}\right)}{1\min}\right) \quad \text{or} \quad \left(\frac{(150ants)}{1\min*1m^2}\right)$$

These are all equivalent mathematical representations of the phenomenon you are describing. Remember, from earlier in this chapter:

$$\frac{A}{B}\ \frac{C}{D} = \frac{A*C}{B*D}$$

$$\frac{1}{\left(\frac{A}{B}\right)} = \frac{B}{A}$$

and, for example, $\frac{\left(\frac{A}{B}\right)}{(C)} = \left(\frac{A}{B}\right)*\frac{1}{(C)} = \frac{A*1}{B*C} = \frac{A}{B*C}$ and $\frac{A}{B*C} = \frac{A*1}{B*C} = \left(\frac{A}{C}\right)*\frac{1}{(B)} = \frac{\left(\frac{A}{C}\right)}{(B)}$

so with these ants, we use the same rules to show:

$$\left(\frac{\left(\frac{150ants}{1\min}\right)}{1m^2}\right) = \left(\frac{150ants}{1\min}\right)*\frac{1}{(1m^2)} = \frac{150ants*1}{1\min*1m^2} = \frac{150ants}{1\min*1m^2} = \left(\frac{150ants}{1m^2}\right)*\frac{1}{(1\min)} = \frac{\left(\frac{150ants}{1m^2}\right)}{1\min}$$

Now what about pitfall #3, the “grouping and flipping fractions” pitfall? Well, imagine you were quickly jotting down all this information about ants, and you said out loud “one hundred fifty ants moved out of the mounds per minute per meter squared,” and you wrote down something like:

$(150ants)/1min/1m^2$ or $\frac{\frac{150ants}{1\min}}{1m^2}$ .....and later you provided this information to someone else, who looked at the expression closely and realized that without parentheses in appropriate places in this bunch of numbers and units, she or he couldn't know whether this represents:

$$\left(\frac{\left(\frac{150ants}{1\min}\right)}{1m^2}\right) \quad \text{or} \quad \left(\frac{150ants}{\left(\frac{1\min}{1m^2}\right)}\right)$$

This is really a problem, because these two expressions are not equivalent! Why? Let's use the same fractions rules again, and see that:

$$\left(\frac{\left(\frac{150ants}{1\min}\right)}{1m^2}\right) = \left(\frac{150ants}{1\min}\right) * \frac{1}{(1m^2)} = \frac{150ants*1}{1\min*1m^2} = \frac{150ants}{1\min*1m^2}$$

BUT

$$\left(\frac{150ants}{\left(\frac{1\min}{1m^2}\right)}\right) = (150ants) * \left(\frac{1}{\left(\frac{1\min}{1m^2}\right)}\right) = (150ants) * \left(\frac{1m^2}{1\min}\right) = \frac{150ants*1m^2}{1\min}$$

which DOES NOT equal

$$\frac{150ants}{1\min*1m^2}$$

Notice how important it is to keep parentheses in the right places, and to follow carefully the rules for how to move the parts of fractions around. When you are solving problems with multiple pieces of information that you need to organize appropriately, these flipping fractions rules become extremely important. Let's look at an example....

Example 7: Back in the chocolate factory, another order arrives the same day as the peanut cluster order. This one is for 10000 boxes of chocolates in celebration of the opening of a store; a local employer wants each of her employees to receive a one pound box of hand-dipped chocolates. You have two months before the order needs to be filled, but you need the chocolates to be made during the week just before they are to be delivered. You know that talented chocolate dippers can hand-dip a chocolate every second, and you know there are 50 chocolates per box. How many dippers are you going to need to hire in order to produce the 1000 boxes of chocolates over the course of a forty hour work week?

Step 1: Write down in shorthand the information you know:

- You need to fill an order for 1000 boxes of chocolates
- There are 50 chocolates per box

- A chocolate dipper can dip one chocolate every second
- You have 40 work hours to fill the order, there are 24 hours in a day, there are 60 minutes in an hour, and there are 60 seconds in a minute.

Step 2: Start putting your information into an equation expressing mathematically the problem you need to solve.

There is quite a bit of information here, so perhaps you can start by simply putting the end result that you want, # of dippers needed, on the right-hand side of the equation:

?*?*?*?*?=?dippers needed

Now you'll being putting information in on the left-hand side, starting with something you know about dippers: one chocolate can be finished per dipper every second. This is beginning to sound like the ants examples used above...you can say the rate of production of chocolates is one chocolate per dipper per second, or you could say one chocolate per second per dipper. You can write, mathematically,

$$\left(\frac{\left(\frac{1\,chocolate}{1\,dipper}\right)}{second}\right) = \frac{1\,chocolate}{1\,dipper * 1\,second} = \frac{\left(\frac{1\,chocolate}{1\,second}\right)}{1\,dipper}$$

These three mathematical representations of what you know are all equivalent; if this doesn't seem right or intuitive to you, go back up to the illustration with ants, and work out the units. With this information in hand, what can you fill into your equation now? You have:

?*?*?*?*?=?dippers needed

and you know something about dippers; you know one dipper produces one chocolate every second. Instead of focusing on chocolates in the numerator (as we just did in the equation above), you want to focus on dippers in the numerator. You need to flip your fractions over!

You want to flip $\frac{\left(\frac{1\,chocolate}{1\,second}\right)}{(1\,dipper)}$ over! How are you going to do this?

If you keep using parentheses, you will be fine. Just flip the fraction over as usual,

putting (1dipper) in the numerator, and $\left(\frac{1\,chocolate}{1\,second}\right)$ in the denominator.

So, you end up with:

$$?*?*?*?*\left(\frac{(1\,dipper)}{\left(\frac{1\,chocolate}{1\,second}\right)}\right) = ?\,dippers$$

Here we have rearranged the representation of the rate of chocolate production correctly so that the unit “dipper” is in the numerator, in order to match with “dipper” on the right-hand side of the equation.

BEWARE! PITFALL!!

This rearrangement of the fraction is a very easy place to make a mistake. Make SURE you keep all the parentheses in the right places! With the parentheses in place, it’s easy to flip this fraction correctly. It’s just a fancy version of:

$$\frac{1}{\left(\frac{A}{B}\right)} = \frac{B}{A}$$ We simply said that $$\frac{1}{\left(\frac{\left(\frac{1\,chocolate}{1\,second}\right)}{(1\,dipper)}\right)} = \frac{(1\,dipper)}{\left(\frac{1\,chocolate}{1\,second}\right)}$$

If you don’t have the parentheses in place, and you try to start flipping fractions around, you may end up confused.....in other words, without parentheses, you may end up wondering what exactly you are dealing with here:

you might see $\frac{1}{\frac{A}{B}}$ and you might wonder whether you’re supposed to be using

$$\frac{1}{\left(\frac{A}{B}\right)} = \frac{B}{A}$$ or $$\left(\frac{\left(\frac{1}{A}\right)}{B}\right) = \frac{1}{A*B}$$ (which does not equal $\frac{B}{A}$) or ...!!?

To see the parallel pattern in the manipulations below, imagine that “A” stands for $\left(\frac{1\,chocolate}{1\,second}\right)$, and “B” stands for $(1\,dipper)$, so setting up the same patterns, you might see:

$$\frac{1}{\left(\frac{\left(\frac{1\,chocolate}{1\sec ond}\right)}{(1\,dipper)}\right)} = \frac{(1\,dipper)}{\left(\frac{1\,chocolate}{1\sec ond}\right)} \quad \text{or} \quad \left(\frac{\left(\frac{1}{\left(\frac{1\,chocolate}{1\sec ond}\right)}\right)}{1\,dipper}\right) = \frac{1}{\left(\frac{1\,chocolate}{1\sec ond}\right)*1\,dipper} \quad \text{or!!?}$$

These are very different fractions! Let's use the correct one for now, and figure out the answer to our question. (Later, we'll come back to using an incorrect one, and see what results from the mistake.)

We have:

$$?*?*?*?*\left(\frac{(1\,dipper)}{\left(\frac{1\,chocolate}{1\sec ond}\right)}\right) = ?\,dippers$$

Let's make this a little easier to look at, rewriting it as:

$$?*?*?*?*\left(\frac{(1\,dipper)*(1\sec ond)}{(1\,chocolate)}\right) = ?\,dippers$$

Step 3: Examine the units in your equation.

Examine your equation again, and start using all the other information you know, matching up units to make them cancel out:

$$\frac{1}{40\,hours}*\left(\frac{1\,hour}{60\min utes}\right)*\left(\frac{1\min ute}{60\sec onds}\right)*1000\,boxes*\left(\frac{50\,chocolates}{1\,box}\right)*\left(\frac{(1\,dipper)*(1\sec ond)}{(1\,chocolate)}\right) = ?\,dippers$$

Cancel out all the units, and you get the unit "dipper" on the left-hand side, matching what you want on the right-hand side of the equation!

$$\frac{1*1*1*1000*50*1\,dipper*1}{40*60*60*1*1} = ?\,dippers$$

That's 0.347 dippers! Of course you can't hire 0.347 dipper! But, you can guess ahead of time that though a talented dipper may be able to dip one chocolate per second at the beginning of the day, by the end of the work-day he might slow down quite a bit! If you hire one dipper for the whole 5 days, he can work at about 1/3 his maximum speed and still get the work done. This seems a safe way to go.

All the examples in this chapter so far have quite general examples that might be encountered by anyone working or traveling around the world. Let's finish this chapter with a much more scientifically-oriented example, one that takes full advantage of what you have learned so far. If you're not biologically-oriented, you can easily skip this example and move on to chapter 3.

Example 8: In plant physiology, it's common to express photosynthetic rates of leaves in units μmol CO_2 *fixed per meter squared leaf each second. If a sunflower leaf is fixing 25 μmol* CO_2 *per meter squared leaf every second, how many hours will it take for a leaf that has area 10centimeter*^{2} *to fix one mol* CO_2*?*

Step 1: Write down in shorthand the information you know, and consider what information you want to end up with:

- The leaf is photosynthesizing 25μmol CO_2 per meter squared leaf into sugar every second
- You wonder how long it will take, in hours, for the leaf to fix two moles CO_2. Given this question, first you need to convert from μmol to moles....you know there are 10^6 μmol in a mol. (To know this, you might have to look up metric prefixes in the back of one of your science books....). Also, you need to change seconds into hours...you can do this easily, via minutes....there are 60 seconds in a minute, and 60 minutes in an hour.
- The leaf's area is 10 cm^2, and the photosynthetic rate is expressed in m^2 . Somehow you need to convert from cm2 leaf to m2 leaf...and you know there are 100 centimeters in one meter.

Step 2: Start putting your information into an equation expressing mathematically the problem you need to solve. This step is going to be slightly harder than in previous examples....there are several confusing components in this calculation.

First, notice what might happen as you try to line up all the information you know on the left-hand side of the equation.....you start by putting in leaf areas and the number of moles CO_2 you are interested in, then you include some conversion factors, the photosynthetic rate, and more conversion factors for time....and you end up with a mess! For example, on the right-hand side of the equation (the side where your answer will end up), you want hours. But take a look at the left-hand side of your equation...hours are in the denominator! This is a first sure sign that you have set up the problem incorrectly.

$$\left(10cm^2\right) * \left(2molCO_2\,fixed\right) * \left(\frac{100^2\,cm^2}{(1m)^2}\right) * \left(\frac{10^6\,\mu molCO_2}{1molCO_2}\right) * \left(\frac{\left(\frac{25\,\mu molCO_2}{m^2}\right)}{\sec}\right) * \left(\frac{60\sec}{1\min}\right) * \left(\frac{60\min}{1hour}\right) \neq ?\ hours$$

Once you notice this, take a minute and try canceling out some more of the units on the left-hand side of this equation....what happens? What units do you end up with?! They most certainly aren't hours....

We'll cancel out what units we can and re-write the equation below....see what you end up with on the left-hand side?!

$$\left(10\,cm^2\right)*(2)*\left(\frac{100^2\,cm^2}{(1m)^2}\right)*\left(\frac{10^6\,\mu molCO_2}{}\right)*\left(\frac{\left(\frac{25\,\mu molCO_2}{m^2}\right)}{}\right)*\left(\frac{60}{1}\right)*\left(\frac{60}{1hour}\right)=$$

$$=\left(\frac{1.8*10^{16}\,cm^4(\mu molCO_2)^2}{m^4\,hour}\right)\neq ?\,hours$$

Something has gone terribly wrong here. Not only are hours in the denominator on the left-hand side of the equation, but also there are a pile of other units that instead of disappearing have multiplied!

Step 3: Examine the units in your equation.

Now that you've noticed that the units as put together in Step 2 just aren't right, start thinking about how to order all the information you know in a way so that hours (and *only* hours) come out on the right-hand side of the equation. The easiest way to start this is to put hours on top on the left-hand side of the equation immediately.....remember,

$$\left(\frac{60\min utes}{1hour}\right)=1=\left(\frac{1hour}{60\min utes}\right)$$ ....and it doesn't matter what version of "1" you multiply by

on the left-hand side of the equation. So, put hours in the numerator .....

$$?*?*?*?*?*?*?*\left(\frac{1hour}{60\min}\right)=?\,hours$$

With this start, now keep filling in your known information on the left-hand side of the equation, keeping in mind that you want to end up with hours (again *only* hours) on the right-hand side when you are done.... What to do first? Notice that you need to get rid of minutes. Look for something in your information that lets you cancel out minutes....and fill it in....

$$?*?*?*?*?*\left(\frac{1\min}{60\sec}\right)*\left(\frac{1hour}{60\min}\right)=?hours$$

...and now you see that you need to get rid of the units "sec"....so add another piece of information, in the correct orientation...

$$?*?*?*?*\left(\frac{1\sec}{\left(\frac{25\,\mu molCO_2}{m^2}\right)}\right)*\left(\frac{1\min}{60\sec}\right)*\left(\frac{1hour}{60\ \min}\right)=?\ hours$$

...and now you need to get rid of μmol CO_2 and m^2, again drawing from the information you have in the problem.... We are going to rearrange the fraction with μmol CO_2 in it to make it easier to see what units need to cancel out where....

$$?*?*\left(\frac{(100\,cm)^2}{(1m)^2}\right)*\left(\frac{10^6\,\mu molCO_2}{1molCO_2}\right)*\left(\frac{m^2*1\sec}{(25\mu molCO_2)}\right)*\left(\frac{1\min}{60\sec}\right)*\left(\frac{1hour}{60\min}\right)=?\ hours$$

If you've kept parentheses in your equation, and everything is going well, now you can sit back for a minute and re-read the problem to remind yourself where you are going. Units are very good at helping you see where you need to go....they are almost like roadmaps leading you to your destination. You want to know how many hours it will take for a leaf to fix 2 moles CO_2. On the left-hand side of the equation, you haven't yet incorporated this information....you need to enter this information, and notice that you have already prepared the way by including the μmol to mol conversion (another fancy version of "1")....

$$?*(2molCO_2\ fixed)*\left(\frac{100^2\,cm^2}{(1m)^2}\right)*\left(\frac{10^6\,\mu molCO_2}{1molCO_2}\right)*\left(\frac{m^2*1\sec}{(25\mu molCO_2)}\right)*\left(\frac{1\min}{60\sec}\right)*\left(\frac{1hour}{60\min}\right)=?\ hours$$

Also, you haven't yet included the leaf area...again, an essential piece of information in solving your problem.....

$$\left(\frac{1}{10cm^2}\right)*(2molCO_2\ fixed)*\left(\frac{100^2\,cm^2}{(1m)^2}\right)*\left(\frac{10^6\,\mu molCO_2}{1molCO_2}\right)*\left(\frac{m^2*1\sec}{(25\mu molCO_2)}\right)*\left(\frac{1\min}{60\sec}\right)*\left(\frac{1hour}{60\min}\right)=?\ hours$$

Now, multiply everything together on the left-hand side of the equation, and see what happens...

You get 22222 hours = ? hours....the units worked out! And WOW, it takes 22222 hours for that little leaf to fix 2 mols CO_2. You've solved the problem!

Now, if you're curious, you want to know how many days that is.....so you figure that out....

$$(22222hours)*\left(\frac{1day}{24hours}\right)=?days=926days$$

Which really makes you wonder about years....

$$(926days)*\left(\frac{1year}{365days}\right)=?years\approx 2.5years$$

Wow! The leaf would have to live for approximately 2.5 years, photosynthesizing 24 hours a day (so must be in a constant environment room at high light all day long and reasonable temperatures...no winters) in order to fix 2 mols CO_2. This is pretty amazing.

You're now well on your way to using units to guide you in answering multi-stage, complex questions.

Chapter 3: Units as informants, and units as spies

3.1 Units as informants, telling you what information you need in order to answer your question....

Sometimes, when you are trying to answer a question that involves many units conversions, you will find yourself wondering whether you have all the information you need to accomplish the task at hand. To find out, you can set up your equations as described in previous chapters of this handbook, arranging what you know in appropriate ways to cancel out units. By setting up your problem as best you can with the information you have, you may find that you are missing some key piece of information. If you're lucky, this may be a simple conversion that you can look up in the back of a book. If, however, you have already finished an economic analysis, or a scientific experiment, or an engineering test, and you don't have all the information you need to answer your question, then you may be out of luck! Scientists, engineers, economists, statisticians, in fact everyone who has to quantify information and analyze data, use scientific units not only to manipulate data once it's gathered, but also help with planning exactly what information needs to be gathered as part of a data set. Whether you are a physics teacher or a homeowner with a green thumb, you can use units to plan your measurements, notes, and calculations, and you can use units to figure out crucial pieces of information. You may find as you are setting up a calculation that what you thought you needed to know at the beginning of the calculation wasn't nearly enough information! You keep having to find more and more information about how particular kinds of units are related to one another. As an example....

Example 9. You work in a nursery focusing on growth of hydroponic tomato plants. These tomato plants are very finicky; they need exactly the right concentration of nutrients in the liquid where their roots are growing in order to produce sweet, juicy tomatoes. You have always used the blue liquid in the corner bucket to refill pots where tomatoes are growing; the nutrient solution in the pots gets low every few days because the tomatoes use so much water. One Friday afternoon, your boss is out "sick," and you find that the corner bucket of refill solution is empty! You have never had to mix this solution up, and you haven't been taught how, but the tomatoes will dry up over the weekend if you don't give them new solution. You read the label on the corner bucket; it says "To make 15 parts per million (by weight) N, put 0.1 oz Plantex into bucket and add water until solution reaches 10 gallons." Relieved, you find a scale, and get ready to measure out 0.1 oz Plantex.....but then you find there are three bags of Plantex, each with different fertilizer mixes! One is 20%N, one is 10% N, and one is 2% N. Which one do you use?

Step 1: Write down in shorthand the information you know, and consider what you want to end up with:

- You will mix 0.1 oz Plantex into 10 gallons water
- You have a choice of three different Plantex mixes: 20%N (by weight), 10% N, and 2%N.
- You want 15 parts per million (by weight) N in the final nutrient solution in the bucket.

Step 2: Start putting your information into an equation expressing mathematically the problem you need to solve.

This is very complicated! Your first reaction might be to write something like this, abbreviating parts per million as ppm to save space on your calculation pad:

?*?*?*?*?*?*?=15ppm N

That's fine! By writing even just this small portion of an equation, you are already on your way to solving it. You know that on the left-hand side of the equation, you're going to have to include information about which fertilizer you choose, how much you add to the bucket, and how much solution you mix up. On the right-hand side, the only unit is ppm N, the final concentration you want to end up with for the tomato plants. So, you know that you're going to have to somehow do some canceling out of gallons, and oz., and somehow you'll have to use %N by weight.

This is a good time to think very carefully to yourself about the units that you are going to have to use. What does 15 parts per million N (by weight) really mean? It means that in every million "parts" of solution, 15 "parts" (by weight) are N. So, for example, for every million oz. of solution, 15 oz. are N. Rephrased, there are 15 oz. N per million oz. of bucket solution. You know how to express this! You can write, using your own subscripts to help you remember which oz are which:

$$?*?*?*? = \frac{15 oz_N}{1{,}000{,}000\, oz_{solution}}$$

You realize that the instructions on the bucket tell you to make 10 gallons of solution, so somehow you'll have to convert from gallons of solution to oz of solution on the left-hand side of the equation above. This may seem impossible at first; gallons are units of volume, and oz. are units of weight! How can those be converted? Here you're going to have to make an assumption. You know the nutrient solution is very light blue, so you know not a lot of the super blue fertilizer is mixed into water to make the final bucket solution. You decide to assume that the bucket solution probably weighs pretty much the same as water, and you know from a "Hummert's Helpful Hints" conversions table next to the balance that one gallon of water weighs 8.3453 pound, and that there are 16 oz. in a pound. So, now you've added some more information to your list of knowns:

- There are 16 oz. in a pound
- One gallon of water weighs 8.3453 one pound

Now what else do you know? You take a look at the bags of fertilizer again, and realize that 20% N by weight means that in every 100 "parts" of fertilizer, 20 "parts" by weight are N. For your three bags of fertilizer, then, you know that 20% N means there are 20 oz N per 100 oz fertilizer, 2% N means there are 2 oz N per 100 oz fertilizer, and 10% N means there are 10 oz N per 100 oz fertilizer. Units are beginning to look familiar....you have oz N when you are thinking about the fertilizer, and also you had oz. N when you were thinking about the final concentration of N in the bucket solution. You know:

- 20% N by weight Plantex fertilizer can be expressed as $\frac{20 oz_N}{100\, oz_{fertilizer}}$
- 2% N by weight fertilizer can be expressed as $\frac{2 oz_N}{100\, oz_{fertilizer}}$
- 20% N by weight fertilizer can be expressed as $\frac{10\, oz_N}{100\, oz_{fertilizer}}$

What else do you know? The instructions on the bucket tell you to put 0.1 oz Plantex fertilizer into the bucket and add water until you have 10 gallons of solution. Even more familiar units are cropping up now! From the label, you have information about how many oz of Plantex fertilizer to use, and also you have just figured out representations of the oz_N per $oz_{fertilizer}$ for each of the Plantex bags. Now you're ready to try to set up an equation, using all the information you know:

$$?*?*?*? = \frac{15 oz_N}{1{,}000{,}000\, oz_{solution}}$$

Start, as usual, by matching units on the left-hand side of the equation with what you want on the right.....perhaps start simply with the weight and volume conversions associated with the solution:

$$?*\left(\frac{1}{10 gallons_{solution}}\right)*\left(\frac{1 gallon_{solution}}{8.3453 lb_{solution}}\right)*\left(\frac{1 lb_{solution}}{16 oz_{solution}}\right) = \frac{15 oz_N}{1{,}000{,}000\, oz_{solution}}$$

In thinking about what to add to the left-hand side of the equation next, look at the units....currently the units on the left-hand side cancel out to $\left(\frac{1}{oz_{solution}}\right)$

But on the right-hand side you want oz_N in the numerator. You need to put in information about the plantex fertilizer you might add. Here's another twist on your usual units problem. On the right-hand side of the equation, you wrote in the final concentration you want in the bucket solution. But it's the Plantex fertilizer bag you don't know about! In previous examples, the "?" that we don't know about has gone on the right-hand side of the equation. Here, we're trying another configuration. This will work fine; you'll just have to note to yourself that you don't know which bag to use. Represent this with a ?:

$$\left(\frac{?\, oz_N}{100 oz_{fertilizer}}\right)*\left(\frac{0.1 oz_{fertilizer}}{10 gallons_{solution}}\right)*\left(\frac{1 gallon_{solution}}{8.3453 lb_{solution}}\right)*\left(\frac{1 lb_{solution}}{16 oz_{solution}}\right) = \frac{15 oz_N}{1{,}000{,}000\, oz_{solution}}$$

Step 3: Examine the units in your equation.

Now try crossing out units. Everything cancels on the left-hand side, except oz_N in the numerator, and $oz_{solution}$ in the denominator. Perfect! But you still have the ? on the left-hand side; whatever the value of the ? is, that will be the bag of fertilizer you use. Cancel out all the units and gather together all the numbers, to get:

$$\left(\frac{?\, oz_N *0.1*1*1}{100*10*8.3453*16 oz_{solution}}\right) = \frac{15 oz_N}{1{,}000{,}000\, oz_{solution}}$$

And then solve for $?oz_N$:

$$?oz_N = \frac{15oz_N}{1{,}000{,}000\,oz_{solution}} * \frac{100*10*8.3453*16oz_{solution}}{0.1*1*1}$$

Turn on your calculator, and multiply out all those numbers on the right-hand side of the equation....they equal 20! So:

$?oz_N=20oz_N$

and therefore ? must equal 20. It is satisfying to finally get to what seems like an answer, but stop for a moment and think about what that "?" is....look back through your notes and calculations...you will find that it's the oz_N per $100oz_{fertilizer}$. . You should use the 20% fertilizer, put 0.1 oz into the bucket, and fill with water up to the 10 gallon mark. The tomatoes are saved!

Now THAT was a units ordeal! But, look at what was accomplished. You used units to help yourself figure out what information you needed from conversion tables in order to solve your problem. You may have been overwhelmed at first by the amount of information in the question, or all the different types of units (volume, weight, ppm, % by weight), but by methodically considering what all the units really meant, and by marching through the units conversions, you were able to figure out exactly the information you needed and how to use it. This is the most exciting part of learning to use scientific units; they are working for you!